ON AN INSTANCE OF THE SMALL COHEN-MACAULAY CONJECTURE II

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Abstract. We show that any d-dimensional local ring A with a dualizing complex, $\operatorname{depth} A = d-1$, and cyclic deficiency module $K^{d-1}(A)$ admits a maximal Cohen–Macaulay module. It is constructed as the unique nonzero cohomology module of the cone of the derived morphism induced by a surjection $A \to K^{d-1}(A)$. This gives new cases of the small Cohen–Macaulay conjecture. When A is quasi-Gorenstein, this module is identified with the first syzygy of the canonical module $\omega_{A/xA}$, for any $x \in \operatorname{ann}_A K^{d-1}(A)$ that is regular on A. This recovers a theorem of Tavanfar and Shimomoto in the 3-dimensional quasi-Gorenstein case with $K^2(A) \cong k$. We also construct examples of section rings satisfying the hypotheses of our theorem.

1. Introduction

The existence of maximal Cohen–Macaulay modules over local rings is a fundamental problem in commutative algebra. While such modules arise naturally over Cohen–Macaulay rings, their existence over non-Cohen–Macaulay rings is far less understood, and relatively few general constructions are available. A result of Tavanfar and Shimomoto [10, Thm. 3.2] provides one such class essentially in dimension 3. Their result reduces to the following statement: if $(A, \mathfrak{m})$ is a 3-dimensional quasi-Gorenstein local ring with $\operatorname{depth} A = 2$ and $H^2_{\mathfrak{m}}(A) \cong k$, then for some nonzerodivisor $x \in \mathfrak{m}$ the first syzygy $\Omega^1\omega_{A/xA}$ of the canonical module of A/xA is maximal Cohen–Macaulay. In our previous work [13], we gave a simpler proof of this result. The proof showed that the hypothesis $H^2_{\mathfrak{m}}(A) \cong k$ enters through the structure of the deficiency module $K^2(A)$, and naturally led to the question of what conditions on $K^{d-1}(A)$ suffice. We show here that cyclicity is sufficient, in arbitrary dimension and without assuming that A is quasi-Gorenstein. This yields new cases of the small Cohen–Macaulay conjecture.

The main result of this paper makes this observation precise.

Theorem 1.1. *Let $(A, \mathfrak{m})$ be a local ring of dimension d admitting a dualizing complex, or equivalently, a homomorphic image of a Gorenstein ring. Suppose that* $\operatorname{depth} A = d-1$ *and that $K^{d-1}(A)$ is cyclic. Then A admits a maximal Cohen–Macaulay module.*

The construction is naturally formulated in the derived category. Let D_A be a dualizing complex normalized so that $H^{-i}(D_A) = K^i(A)$, and set $X = D_A[d-1]$. Then $H^{-1}(X) = \omega_A$ and $H^0(X) = K^{d-1}(A)$. A generator of the cyclic module $K^{d-1}(A)$ determines a morphism $f\colon A \to X$ in $D(A)$, and we consider the module $M := H^{-1}(\operatorname{Cone}(f))$. The surjectivity of the induced map $A \to K^{d-1}(A)$ implies that the cone has cohomology concentrated in degree -1. Using duality, we show that the dual of this cone is again, up to shift, the cone associated to another generator of $K^{d-1}(A)$, and hence also has cohomology concentrated in degree -1. Local duality then implies M is maximal Cohen–Macaulay.

Although the proof can be presented succinctly in the derived setting, we also give a concrete description of the resulting module M. In the setup of Theorem 1.1, suppose that $x \in \operatorname{ann}_A K^{d-1}(A)$ is a nonzerodivisor on A. We show in Lemma 2.2 that M is the kernel of a natural map

$$\Phi\colon \omega_A \oplus A \longrightarrow \omega_{A/xA}.$$

Moreover, when A is quasi-Gorenstein, namely when $\omega_A \cong A$, we show that M is precisely the first syzygy $\Omega^1\omega_{A/xA}$ of the canonical module of A/xA, see Proposition 2.5. Thus our construction recovers the 3-dimensional construction of Tavanfar and Shimomoto and extends it to arbitrary dimension under the cyclicity hypothesis on $K^{d-1}(A)$. Since $\omega_{A/xA}$ is a Cohen–Macaulay module of dimension $d-1$, this fits naturally into a more general question about when higher syzygies

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of lower-dimensional Cohen–Macaulay modules are maximal Cohen–Macaulay. In Section 2.2, we briefly discuss a condition for the r-th syzygy of a $d-r$-dimensional Cohen–Macaulay module to be maximal Cohen–Macaulay.

Finally, we give examples of families of section rings satisfying the hypotheses of Theorem 1.1, including some rings whose completions admit no small Cohen–Macaulay algebra.

Notation. Throughout the paper, "local ring" means Noetherian local ring. We use the standard convention for cohomologically indexed complexes as in Weibel [12]: for a complex C, its p-th translate $C[p]$ is defined by $C[p]^n = C^{n-p}$, with differential $d^n_{C[p]} = (-1)^p d^{n-p}_C$. Let $D(A)$ denote the derived category of A-modules, and let $D^b_c(A)$ denote the full subcategory of $D(A)$ consisting of complexes with bounded, finitely generated cohomology.

By [7, Cor. 1.4], a local ring admits a dualizing complex if and only if it is a homomorphic image of a Gorenstein local ring; we use these equivalent conditions interchangeably. If A is a homomorphic image of a Gorenstein local ring R of dimension n, then for a finitely generated A-module M we write $K^i(M) := \operatorname{Ext}^{n-i}_R(M,R)$ for the i-th deficiency module of M.

For a local ring $(A,\mathfrak{m},k)$, we denote by $(-)^\vee$ the Matlis dual functor $(-)^\vee := \operatorname{Hom}_A(-,E_A(k))$, where $E_A(k)$ is the injective hull of k over A. For a k-vector space V, we write V' for its k-linear dual.

2. The Cone Construction

Proof of Theorem 1.1. Let D_A denote a dualizing complex for A, normalized so that $H^{-i}(D_A) = K^i(A)$; see [9, Sec. 3]. Set

$$X := D_A[d-1].$$

Let ω_A denote the canonical module of A. Since $\operatorname{depth} A = d-1$, $K^i(A) = 0$ for $i \notin \{d-1, d\}$, therefore

$$H^{-1}(X) = \omega_A, \qquad H^0(X) = K^{d-1}(A), \qquad H^i(X) = 0 \quad \text{for } i \notin \{-1,0\}.$$

Denote $C := K^{d-1}(A)$. Since

$$\operatorname{Hom}_{D(A)}(A,X) = H^0(\mathbf{R}\operatorname{Hom}_A(A,X)) \cong H^0(X) = C,$$

a generator $c \in C$ determines a morphism $f\colon A \to X$ in $D(A)$. The induced map on H^0, $H^0(f)\colon A \to C$, is given by multiplication by c which is therefore surjective. There is an exact triangle

$$A \xrightarrow{f} X \longrightarrow \operatorname{Cone}(f) \longrightarrow A[-1]. \tag{2.1}$$

The associated long exact sequence in cohomology contains the exact sequence

$$0 \longrightarrow \omega_A \longrightarrow H^{-1}(\operatorname{Cone}(f)) \longrightarrow A \xrightarrow{H^0(f)} C \longrightarrow H^0(\operatorname{Cone}(f)) \longrightarrow 0.$$

Since $H^0(f)\colon A \to C$ is surjective, we have $H^0(\operatorname{Cone}(f)) = 0$. Hence $\operatorname{Cone}(f)$ has cohomology concentrated in degree -1, so

$$\operatorname{Cone}(f) \simeq M[-1], \qquad M := H^{-1}(\operatorname{Cone}(f)).$$

Moreover, the above long exact sequence yields a short exact sequence

$$0 \longrightarrow \omega_A \longrightarrow M \longrightarrow \operatorname{Ann}_A(C) \longrightarrow 0.$$

Define

$$\mathbb{D}(-) := \mathbf{R}\operatorname{Hom}_A(-,D_A)[d-1].$$

By [5, Prop. V.2.1] or [11, Tag 0A7C], the functor $\mathbb{D}$ is a contravariant equivalence on $D^b_c(A)$ and satisfies biduality $\mathbb{D}^2 \simeq \operatorname{id}$. By construction,

$$\mathbb{D}(A) = X \qquad \text{and} \qquad \mathbb{D}(X) \simeq A.$$

Let $\delta_A\colon A \xrightarrow{\simeq} \mathbb{D}^2(A) = \mathbb{D}(X)$ be the biduality isomorphism. For $f \in \operatorname{Hom}_{D(A)}(A,X)$, define its transpose by

$$f^\dagger\colon A \xrightarrow{\delta_A} \mathbb{D}(X) \xrightarrow{\mathbb{D}(f)} \mathbb{D}(A) = X.$$

Thus we obtain a map

$$\tau\colon \operatorname{Hom}_{D(A)}(A, X) \longrightarrow \operatorname{Hom}_{D(A)}(A, X), \qquad f \longmapsto f^{\dagger}.$$

Since $\mathbb{D}$ is a contravariant equivalence, τ is a bijection. Moreover, τ is A-linear. Thus, under the natural isomorphism $\operatorname{Hom}_{D(A)}(A, X) \cong H^0(X) = C$, the operation $f \mapsto f^{\dagger}$ induces an A-module automorphism of C. Therefore, $f^{\dagger}$ corresponds to another generator of C, hence $H^0(f^{\dagger})\colon A \to C$ is also surjective. By the same argument as above, $\operatorname{Cone}(f^{\dagger})$ has cohomology concentrated in degree -1. Hence

$$\operatorname{Cone}(f^{\dagger}) \simeq N[-1], \qquad N := H^{-1}(\operatorname{Cone}(f^{\dagger})).$$

Applying $\mathbb{D}$ to the exact triangle (2.1) gives the following exact triangle after rotations

$$A \xrightarrow{f^{\dagger}} X \longrightarrow \mathbb{D}(M) \longrightarrow A[-1].$$

Therefore,

$$\mathbb{D}(M) \simeq \operatorname{Cone}(f^{\dagger}) \simeq N[-1],$$

and hence

$$\mathbf{R}\operatorname{Hom}_A(M, D_A) \simeq N[-d].$$

By local duality [5, Thm. V.6.2], we have

$$H^i_{\mathfrak{m}}(M) \cong H^{-i}(\mathbf{R}\operatorname{Hom}_A(M, D_A))^{\vee} \cong \begin{cases} N^{\vee}, & i = d, \\ 0, & i \neq d. \end{cases}$$

It follows that M is a maximal Cohen–Macaulay A-module. □

Remark 2.1. The deficiency module $K^{d-1}(A)$ is also related to the non-S_2 locus. In the setup of Theorem 1.1, suppose additionally that A is equidimensional and unmixed and that its non-S_2 locus is nonempty. Set $B := \operatorname{End}_A(\omega_A)$, $C := B/A$. Then $\dim C = d - 2$, as in the proof of [14, Thm. 4.7], and the connecting morphism $K^{d-1}(A) \to K^{d-2}(C)$ is the S_2-hull map by [14, Thm. 4.3]. Thus, if $K^{d-1}(A)$ is moreover equidimensional and S_2, then its cyclicity implies that the top-dimensional part of the non-S_2 locus is connected in codimension one by [14, Thm. 3.2].

2.1. **Identification of the cone construction.** We now give a more explicit description of the cone module constructed above. We begin with the following lemma, which identifies the cone module with the kernel of a natural map.

Lemma 2.2. *Let A be a commutative ring, and let $X \in D(A)$ with*

$$H^{-1}(X) = U, \qquad H^0(X) = V, \qquad H^i(X) = 0 \quad \text{for } i \notin \{-1, 0\},$$

where U and V are finitely generated A-modules. Let $\alpha\colon F \twoheadrightarrow V$ be a surjection with F a projective A-module. Suppose that $x \in A$ is a nonzerodivisor on both A and U, and that $xV = 0$. Set

$$B := A/xA, \qquad W := H^0(\mathbf{R}\operatorname{Hom}_A(B, X))\,.$$

Since F is projective, there is a natural isomorphism $\operatorname{Hom}_{D(A)}(F, X) \cong \operatorname{Hom}_A(F, V)$. Let $f\colon F \to X$ be the morphism in $D(A)$ corresponding to α, and set $M_f := H^{-1}(\operatorname{Cone}(f))$. Then the following hold.

(1) *There is a canonical short exact sequence*

$$0 \longrightarrow U/xU \xrightarrow{\overline{\tau}} W \xrightarrow{\pi} V \longrightarrow 0.$$

(2) *There is a lift $\widetilde{\alpha}\colon F \to W$ of α, so that $\pi \circ \widetilde{\alpha} = \alpha$ and the following holds. Define*

$$\Phi\colon U \oplus F \longrightarrow W, \qquad \Phi(u, z) = \overline{\tau}(\overline{u}) + \widetilde{\alpha}(z),$$

where $\overline{u}$ denotes the image of u in U/xU. Then Φ is surjective, and $M_f \cong \ker(\Phi)$. Moreover, M_f fits into a short exact sequence

$$0 \longrightarrow U \longrightarrow M_f \longrightarrow \ker(\alpha) \longrightarrow 0.$$

Proof. By taking a good truncation of X, see [12, 1.2.7, p. 9], we may represent X by a two-term complex $X = [P \xrightarrow{\delta} Q]$, where P is placed in degree -1 and Q in degree 0. Thus

$$U = \ker(\delta), \qquad V = \operatorname{coker}(\delta).$$

Since F is projective, we may lift $\alpha\colon F \to V$ to a map $q\colon F \to Q$ which represents the derived morphism $f\colon F \to X$ corresponding to α. Hence

$$M_f := H^{-1}(\operatorname{Cone}(f)) = \{(p,z) \in P \oplus F \mid \delta(p) - q(z) = 0\}\,. \tag{2.2}$$

Moreover, since α is surjective and $V = \operatorname{coker}(\delta) = Q/\operatorname{im}(\delta)$, we have $Q = \operatorname{im}(q) + \operatorname{im}(\delta)$, and hence $H^0(\operatorname{Cone}(f)) = 0$. Thus $\operatorname{Cone}(f) \simeq M_f[-1]$. Projection onto the second factor F induces a surjection $M_f \twoheadrightarrow \ker(\alpha)$ whose kernel is naturally identified with $U = \ker \delta$. Hence, we have an exact sequence

$$0 \longrightarrow U \longrightarrow M_f \longrightarrow \ker(\alpha) \longrightarrow 0.$$

Since $xV = 0$, we have $xq(F) \subseteq \operatorname{im}(\delta)$. Since F is projective, the map $xq\colon F \to \operatorname{im}(\delta)$ lifts through the surjection $\delta\colon P \twoheadrightarrow \operatorname{im}(\delta)$. Thus there exists a map $t\colon F \to P$ such that

$$\delta \circ t = xq. \tag{2.3}$$

Indeed, this t is precisely a chain homotopy from xf to 0. Applying the contravariant functor $\mathbf{R}\operatorname{Hom}_A(-, X)$ to the exact triangle $A \xrightarrow{x} A \to B \to A[-1]$ yields the exact triangle

$$\mathbf{R}\mathrm{Hom}_A(B,X) \to \mathbf{R}\mathrm{Hom}_A(A,X) \xrightarrow{x} \mathbf{R}\mathrm{Hom}_A(A,X) \to \mathbf{R}\mathrm{Hom}_A(B,X)[-1]. \tag{2.4}$$

Since $\mathbf{R}\mathrm{Hom}_A(A,X) \simeq X$, we obtain

$$\operatorname{Cone}(x\colon X \to X) \simeq \mathbf{R}\mathrm{Hom}_A(B,X)[-1].$$

The cone of $x\colon X \to X$ is represented by the complex

$$0 \longrightarrow P \xrightarrow{d^{-2}} Q \oplus P \xrightarrow{d^{-1}} Q \longrightarrow 0,$$

where

$$d^{-2}(r) = (-\delta r, -xr) \qquad \text{and} \qquad d^{-1}(v,s) = -xv + \delta s.$$

Therefore,

$$W = H^0\mathbf{R}\mathrm{Hom}_A(B,X) = H^{-1}\operatorname{Cone}(x\colon X \to X) \cong \frac{\{(s,v) \in P \oplus Q : \delta(s) - xv = 0\}}{\{(xr, \delta r) : r \in P\}}.$$

The long exact sequence of cohomology associated to the exact triangle (2.4) contains

$$H^{-1}(X) \xrightarrow{x} H^{-1}(X) \xrightarrow{\tau} H^0\mathbf{R}\mathrm{Hom}_A(B,X) \xrightarrow{\pi} H^0(X) \xrightarrow{x} H^0(X).$$

This yields the short exact sequence

$$0 \longrightarrow U/xU \xrightarrow{\overline{\tau}} W \xrightarrow{\pi} V \longrightarrow 0. \tag{2.5}$$

Under the above description of W, the maps τ and π are given by

$$\tau(u) = [(u,0)] \qquad \text{and} \qquad \pi([(s,v)]) = \overline{v},$$

where $\overline{v}$ denotes the image of $v \in Q$ in $V = Q/\operatorname{im}\delta$. Indeed, under the identification $\mathbf{R}\mathrm{Hom}_A(B,X) \simeq \operatorname{Cone}(x\colon X \to X)[-1]$, these maps are induced by the canonical inclusion and projection in the short exact sequence of complexes

$$0 \longrightarrow X \longrightarrow \operatorname{Cone}(x\colon X \to X) \longrightarrow X[-1] \longrightarrow 0,$$

see [12, 1.5.2, p. 19 and p. 371].

For $z \in F$, by (2.3), we have $\delta(t(z)) = xq(z)$. Hence $(t(z), q(z))$ is a (-1)-cycle in $\operatorname{Cone}(x\colon X \to X)$. Define

$$\widetilde{\alpha}\colon F \longrightarrow W, \qquad \widetilde{\alpha}(z) = [(t(z), q(z))].$$

Then $\pi(\widetilde{\alpha}(z)) = [q(z)] = \alpha(z)$, so $\widetilde{\alpha}$ is indeed a lift of α. Then $\Phi\colon U \oplus F \longrightarrow W$, $\Phi(u,z) = \overline{\tau}(\overline{u}) + \widetilde{\alpha}(z)$ is given by $\Phi(u,z) = [(u + t(z), q(z))]$. Using the exact sequence (2.5) and that $\alpha = \pi \circ \widetilde{\alpha}$ is surjective, it follows that Φ is surjective.

Define the homomorphism

$$\Theta\colon M_f \longrightarrow U \oplus F, \qquad \Theta(p,z) = (xp - t(z), z).$$

First, Θ is well defined. Indeed, for $(p,z) \in M_f$, by (2.2) and (2.3), we have

$$\delta(xp - t(z)) = x\delta(p) - \delta(t(z)) = xq(z) - xq(z) = 0.$$

Hence $xp - t(z) \in \ker\delta = U$. Next,

$$\Phi \circ \Theta(p,z) = \Phi(xp - t(z), z) = [(xp, q(z))] = [(xp, \delta(p))].$$

Since $(xp, \delta(p))$ is a boundary in $\operatorname{Cone}(x\colon X \to X)$, it follows that $\Phi \circ \Theta = 0$. Thus $\operatorname{im}\Theta \subseteq \ker\Phi$.

We next prove that Θ is injective. Suppose that $\Theta(p,z) = 0$. Then $z = 0$ and $xp = 0$. Since $(p,0) \in M_f$, (2.2) gives $\delta(p) = 0$, so $p \in U$. As x is regular on U, we have $p = 0$.

Finally, we show that Θ is surjective onto $\ker\Phi$. Let $(u,z) \in \ker\Phi$. Then $[(u + t(z), q(z))] = 0$ in W. Hence there exists $p \in P$ such that $u + t(z) = xp$, $q(z) = \delta(p)$. It follows that $(p,z) \in M_f$, and $\Theta(p,z) = (xp - t(z), z) = (u,z)$.

Therefore, $\Theta\colon M_f \xrightarrow{\sim} \ker\Phi$ is an isomorphism. This finishes the proof. □

Remark 2.3. If x is not regular on U, the same construction yields an exact sequence

$$0 \longrightarrow (0 :_U x) \longrightarrow M_f \xrightarrow{\Theta} \ker\Phi \longrightarrow 0.$$

In [10], Tavanfar and Shimomoto proved a case of the existence of maximal Cohen–Macaulay modules which, in essence, is the following three-dimensional result.

Theorem 2.4 ([10, Thm. 3.2])**.** *Let $(A,\mathfrak{m})$ be a quasi-Gorenstein local ring which is a homomorphic image of a Gorenstein local ring of dimension 3. Suppose that* $\operatorname{depth} A = 2$ *and $H^2_{\mathfrak{m}}(A) \cong k$. Then, for some nonzerodivisor $x \in \mathfrak{m}$, the first syzygy module $\Omega^1\omega_{A/xA}$ of the canonical module $\omega_{A/xA}$ of A/xA is a maximal Cohen–Macaulay module.*

The proof in [10] is somewhat involved. In [13], we give a simpler proof of this result.

Using Lemma 2.2, we can extend this result and show that, the cone construction in Theorem 1.1 is naturally identified with the first syzygy module $\Omega^1\omega_{A/xA}$ whenever A is quasi-Gorenstein,

Proposition 2.5. *Let $(A,\mathfrak{m})$ be a quasi-Gorenstein local ring of dimension d which is a homomorphic image of a Gorenstein local ring. Suppose that* $\operatorname{depth} A = d-1$ *and that $K^{d-1}(A)$ is cyclic. Let $x \in \operatorname{ann}_A K^{d-1}(A)$ be a nonzerodivisor on A. Then the first syzygy module $\Omega^1\omega_{A/xA}$ is a maximal Cohen–Macaulay A-module.*

Proof. Since A is quasi-Gorenstein, we have $\omega_A \cong A$. The complex X appearing in the proof of Theorem 1.1 then satisfies $H^{-1}(X) = \omega_A \cong A$, $H^0(X) = K^{d-1}(A)$, and $H^i(X) = 0$, $i \notin \{-1, 0\}$. Moreover, $H^0\mathbf{R}\mathrm{Hom}_A(B,X) \cong \omega_B$, where $B = A/xA$. Hence, by Lemma 2.2, there is a short exact sequence

$$0 \longrightarrow A/xA \xrightarrow{\overline{\tau}} \omega_B \longrightarrow K^{d-1}(A) \longrightarrow 0. \tag{2.6}$$

Let $\alpha\colon A \to K^{d-1}(A)$ be a surjection, and let $\widetilde{\alpha}\colon A \to \omega_B$ be the lift given by Lemma 2.2. Then the map $\Phi\colon A^2 \to \omega_B$, $\Phi(u,z) = \overline{\tau}(\overline{u}) + \widetilde{\alpha}(z)$, is surjective, and Lemma 2.2 gives $M_f := H^{-1}(\operatorname{Cone}(f)) \cong \ker\Phi$. By Theorem 1.1, M_f is maximal Cohen–Macaulay.

It remains to show that M_f is the first syzygy module of ω_B. It suffices to show that ω_B is minimally generated by two elements. Suppose otherwise that ω_B is generated by one element, and hence is cyclic. Since $B \hookrightarrow \omega_B$, it follows that $\omega_B \cong B$. Hence (2.6) gives $K^{d-1}(A) \cong B/aB$, for some nonzerodivisor $a \in B$, and therefore $\dim K^{d-1}(A) = \dim B - 1 = d-2$. On the other hand, since $A \cong \omega_A$ is S_2, by [9, Prop. 3.1], $\dim K^{d-1}(A) \le d-3$, a contradiction. Therefore, ω_B is minimally generated by two elements and $M_f \cong \ker\Phi \cong \Omega^1\omega_B$. □

2.2. **Higher syzygies of lower-dimensional Cohen–Macaulay modules.** In the setup of Proposition 2.5, the exact sequence

$$0 \longrightarrow M_f \longrightarrow A^2 \longrightarrow \omega_B \longrightarrow 0$$

gives $\operatorname{depth}_A \omega_B \ge \min\{\operatorname{depth}_A M_f - 1, \operatorname{depth} A\} \ge d-1$. Hence ω_B is a Cohen–Macaulay A-module of dimension $d-1$.

There are many natural sources of Cohen–Macaulay modules of lower dimension. For example, in the introduction of our previous work [13], we recorded the fact that if $(A,\mathfrak{m})$ is quasi-Gorenstein and is a homomorphic image of a Gorenstein local ring of dimension d, with $\operatorname{depth} A = d-1$, then $K^{d-1}(A)$ is Cohen–Macaulay of dimension $d-3$. Another example is obtained by taking a quotient of A of dimension 2; its S_2-ification is then a Cohen–Macaulay module of dimension 2. The condition below isolates the obstruction to obtaining a maximal Cohen–Macaulay module from a higher syzygy, but it does not appear to hold in general for such lower-dimensional Cohen–Macaulay modules and therefore does not seem to yield a broader construction of maximal Cohen–Macaulay modules.

Proposition 2.6. *Let A be a local ring of dimension d which is a homomorphic image of a Gorenstein local ring. Suppose* $\operatorname{depth} A = d-1$, *and let L be a finitely generated Cohen–Macaulay A-module of dimension $d-r$, $r \geq 1$. Then* $\operatorname{depth} \Omega^i L \geq d-1$ *for every $i \geq r-1$. Moreover, $\Omega^r L$ is maximal Cohen–Macaulay if and only if the natural map*

$$K^{d-1}(\Omega^{r-1}L) \to K^{d-1}(A)^{\beta_{r-1}}$$

is surjective, where β_i denotes the i-th Betti number of L.

Proof. Denote by $Z_i := \Omega^i L$ the i-th syzygy of L. Let $F_\bullet$ be a minimal free resolution of L, with $F_i = A^{\beta_i}$. For each $i \geq 0$, we have a short exact sequence

$$0 \longrightarrow Z_{i+1} \longrightarrow F_i \longrightarrow Z_i \longrightarrow 0. \tag{2.7}$$

Then, $\operatorname{depth} Z_{i+1} \geq \min\{\operatorname{depth} F_i, \operatorname{depth} Z_i + 1\} = \min\{d-1, \operatorname{depth} Z_i + 1\}$. Since $\operatorname{depth} Z_0 = \operatorname{depth} L = d-r$, we have $\operatorname{depth} Z_i \geq d-1$ for all $i \geq r-1$.

The long exact sequence of local cohomology associated to (2.7) with $i = r-1$ contains

$$0 \longrightarrow H^{d-1}_{\mathfrak{m}}(Z_r) \longrightarrow H^{d-1}_{\mathfrak{m}}(F_{r-1}) \longrightarrow H^{d-1}_{\mathfrak{m}}(Z_{r-1}).$$

So Z_r is maximal Cohen–Macaulay if and only if the natural map $H^{d-1}_{\mathfrak{m}}(F_{r-1}) \to H^{d-1}_{\mathfrak{m}}(Z_{r-1})$ is injective. By local duality, this is equivalent to the surjectivity of $\widehat{K^{d-1}(Z_{r-1})} \to \widehat{K^{d-1}(F_{r-1})}$. which is equivalent to the surjectivity of $K^{d-1}(Z_{r-1}) \to K^{d-1}(F_{r-1}) = K^{d-1}(A)^{\beta_{r-1}}$ by faithful flatness of completion. ☐

Corollary 2.7. *In the setup of Theorem 2.4, the first syzygy $\Omega^1\omega_{A/xA}$ is maximal Cohen–Macaulay.*

Proof. The A-module $\omega_{A/xA}$ is Cohen–Macaulay of dimension 2. By Proposition 2.6, it is enough to show that the natural map $K^2(\omega_{A/xA}) \to K^2(A)^2$ is surjective. This is precisely the map

$$f\colon \operatorname{Ext}^1_R(\omega_{A/xA}, R) \longrightarrow \operatorname{Ext}^1_R(A^2, R) \cong k^2,$$

which was shown to be surjective in [13, p. 5]. ☐

3. Examples of Section Rings

In this section, we give some examples of section rings to which our main theorem applies.

Proposition 3.1. *Let k be a field, and let $r, m \geq 1$. Let Z be a projective k-variety of dimension r satisfying*

$$H^i(Z, \mathcal{O}_Z) \cong \begin{cases} k, & i = 0, r, \\ 0, & \textit{otherwise}. \end{cases} \tag{3.1}$$

Let S be a projective k-variety of dimension m satisfying

$$H^j(S, \mathcal{O}_S) \cong \begin{cases} k, & j = 0, \\ 0, & j > 0. \end{cases} \tag{3.2}$$

Set $X_0 = Z \times_k S$. For $1 \leq j \leq m-1$, suppose that $X_j \subset X_{j-1}$ is an effective Cartier divisor and that

$$H^q(X_{j-1}, \mathcal{O}_{X_{j-1}}(-X_j)) = 0 \qquad \textit{for all } q < \dim X_{j-1}. \tag{3.3}$$

If $m = 1$, set $X = X_0$. If $m \geq 2$, set $X = X_{m-1}$. Assume that X is integral. Let L be an ample line bundle on X satisfying

$$H^i(X, L^n) = 0 \qquad \textit{for all } n \neq 0 \textit{ and } 1 \leq i \leq r. \tag{3.4}$$

Define the section ring

$$R = R(X, L) = \bigoplus_{n\geq 0} H^0(X, L^n),$$

let $\mathfrak{m} = R_+$*, and set* $A = R_{\mathfrak{m}}$*. Then* $\dim A = r+2$, $\operatorname{depth} A = r+1$, *and* $H^{r+1}_{\mathfrak{m}A}(A) \cong k$.

Proof. First, we have $\dim X_0 = \dim(Z \times_k S) = r + m$, and $\dim X = \dim X_0 - (m-1) = r+1$.

By the Künneth formula,

$$H^i(X_0, \mathcal{O}_{X_0}) \cong \bigoplus_{a+b=i} H^a(Z, \mathcal{O}_Z) \otimes_k H^b(S, \mathcal{O}_S).$$

Thus by (3.1) and (3.2), we have

$$H^i(X_0, \mathcal{O}_{X_0}) \cong H^i(Z, \mathcal{O}_Z) \cong \begin{cases} k, & i = 0, r, \\ 0, & \text{otherwise.} \end{cases}$$

For each j, there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{X_{j-1}}(-X_j) \longrightarrow \mathcal{O}_{X_{j-1}} \longrightarrow \mathcal{O}_{X_j} \longrightarrow 0.$$

The associated long exact sequence in cohomology contains

$$H^i(X_{j-1}, \mathcal{O}_{X_{j-1}}(-X_j)) \longrightarrow H^i(X_{j-1}, \mathcal{O}_{X_{j-1}}) \longrightarrow H^i(X_j, \mathcal{O}_{X_j}) \longrightarrow H^{i+1}(X_{j-1}, \mathcal{O}_{X_{j-1}}(-X_j)).$$

By (3.3), $H^i(X_{j-1}, \mathcal{O}_{X_{j-1}}) \cong H^i(X_j, \mathcal{O}_{X_j})$ for every $i \leq \dim X_{j-1} - 2$. Since $j \leq m-1$, we have $\dim X_{j-1} \geq r+2$. Therefore, iterating these isomorphisms gives

$$H^i(X, \mathcal{O}_X) \cong H^i(X_0, \mathcal{O}_{X_0}) \qquad \text{for } 0 \leq i \leq r. \tag{3.5}$$

Since X is integral and L is ample, we have $H^0(X, L^j) = 0$ for $j < 0$. Moreover, the section ring

$$R = R(X, L) := \bigoplus_{n\geq 0} H^0(X, L^n)$$

is a finitely generated graded domain over k, with $\operatorname{Proj} R \cong X$ and $\widetilde{R(n)} \cong L^n$, $n \in \mathbb{Z}$, see [11, Tag 0B5T] and [3, Thm. 4.5.2, Prop. 4.5.5]. Hence $\dim A = \dim R = \dim X + 1 = r+2$. By the Serre–Grothendieck correspondence [2, Thm. 20.4.4], for every $i \geq 1$,

$$H^{i+1}_{R_+}(R) \cong \bigoplus_{j\in\mathbb{Z}} H^i(X, L^j).$$

By (3.4) and (3.5), $H^{i+1}_{R_+}(R) = 0$ for $1 \leq i \leq r-1$, and $H^{r+1}_{R_+}(R) \cong k$. From the exact sequence [2, Thm. 2.2.6(c)]

$$0 \longrightarrow H^0_{R_+}(R) \longrightarrow R \longrightarrow \bigoplus_{j\in\mathbb{Z}} H^0(X, L^j) \longrightarrow H^1_{R_+}(R) \longrightarrow 0,$$

the middle map is an isomorphism. Hence $H^0_{R_+}(R) = H^1_{R_+}(R) = 0$. Localizing at the homogeneous maximal ideal $\mathfrak{m} = R_+$ and using local duality give the desired result. ☐

The conditions of Proposition 3.1 can be constructed using Kodaira vanishing. For example, one may take Z to be a Calabi–Yau variety over a field k of characteristic 0 and $S = \mathbb{P}^m_k$. Throughout this section, we call a smooth geometrically connected projective variety Z over k a *projective Calabi–Yau variety* if

$$\omega_Z \cong \mathcal{O}_Z \qquad \text{and} \qquad H^i(Z, \mathcal{O}_Z) = 0 \quad \text{for } 0 < i < \dim Z.$$

Corollary 3.2. *Let* k *be a field of characteristic* 0*, and let* $r, m \geq 1$*. Let* Z *be a projective Calabi–Yau variety of dimension* r *over* k*. Let* S *be a smooth projective geometrically connected variety of dimension* m *satisfying*

$$H^j(S, \mathcal{O}_S) = 0 \qquad \text{for all } j > 0.$$

Let $X_0 = Z \times S$*. For* $1 \leq j \leq m-1$*, let* $X_j \subset X_{j-1}$ *be a smooth ample effective Cartier divisor, and set* $X = X_{m-1}$*. Assume that* ω_X *is ample. Let* $a \geq 2$*, and set* $L = \omega_X^{\otimes a}$*. Then the local section ring*

$$A = \Big(\bigoplus_{n\geq 0} H^0(X, \omega_X^{\otimes an})\Big)_{R_+}$$

satisfies

$$\dim A = r+2, \qquad \operatorname{depth} A = r+1, \qquad K^{r+1}(A) \cong k.$$

Proof. We check the conditions in Proposition 3.1 are satisfied. Since Z is geometrically connected, $H^0(Z, \mathcal{O}_Z) \cong k$ by [11, Tag 0FD2]. Since $\omega_Z \cong \mathcal{O}_Z$, Serre duality [6, Cor. 7.7] gives $H^r(Z, \mathcal{O}_Z)' \cong H^0(Z, \omega_Z) \cong k$. Together with the assumed vanishing $H^i(Z, \mathcal{O}_Z) = 0$ for $0 < i < r$ shows that Z satisfies (3.1). Similarly, S satisfies (3.2).

For each j, the line bundle $\mathcal{O}_{X_{j-1}}(X_j)$ is ample. By Serre duality,

$$H^q(X_{j-1}, \mathcal{O}_{X_{j-1}}(-X_j))' \cong H^{\dim X_{j-1}-q}(X_{j-1}, \omega_{X_{j-1}} \otimes \mathcal{O}_{X_{j-1}}(X_j)).$$

By Kodaira vanishing in characteristic 0, see [4, Cor. 2.11], the latter vanishes whenever $q < \dim X_{j-1}$. Hence (3.3) is satisfied.

Now let $n > 0$. Since $L^n = \omega_X^{\otimes an} = \omega_X \otimes \omega_X^{\otimes(an-1)}$ and $a \geq 2$, the line bundle $\omega_X^{\otimes(an-1)}$ is ample. Kodaira vanishing therefore gives $H^i(X, L^n) = 0$ for all $i > 0$.

For $n < 0$, since $\dim X = r+1$, by Serre duality and Kodaira vanishing applied to the ample line bundle $L^{-n} = \omega_X^{\otimes(-an)}$ gives

$$H^i(X, L^n)' \cong H^{r+1-i}(X, \omega_X \otimes L^{-n}) = 0 \qquad \text{whenever } r+1-i > 0.$$

Thus $H^i(X, L^n) = 0$ for $i < r+1, n \neq 0$. Therefore (3.4) is satisfied for L. Finally, the argument proving (3.5) gives $H^0(X, \mathcal{O}_X) \cong k$ thus X is geometrically connected by [11, Tag 0FD2]. Moreover, since X is smooth and geometrically connected, it is geometrically integral. Therefore all the hypotheses of Proposition 3.1 are satisfied. □

Example 3.1. Let Z be a projective Calabi–Yau variety over a field k of characteristic 0, and let M be an ample line bundle on Z. Set $Y := Z \times \mathbb{P}^2_k$, and write $H := p_1^*M$ and $F := p_2^*\mathcal{O}_{\mathbb{P}^2_k}(1)$, where p_1 and p_2 are the two projections. Choose integers $c \gg 0$ and $b > 3$, and let $X \in |cH + bF|$ be a smooth divisor. Since $\omega_Y \cong p_1^*\omega_Z \otimes p_2^*\omega_{\mathbb{P}^2_k} \cong \mathcal{O}_Y(-3F)$, the adjunction formula gives $\omega_X \cong \mathcal{O}_X(cH + (b-3)F)$. Since $c > 0$ and $b - 3 > 0$, the line bundle ω_X is ample. For any $a \geq 2$, set $L := \omega_X^{\otimes a}$. Then X and L satisfy the hypotheses of Corollary 3.2.

Moreover, using the criterion of Bhatt [1, Thm. 1.3], one can obtain classes of rings in characteristic p whose completions admit no small Cohen–Macaulay algebra, but which satisfy Proposition 3.1 and hence admit a maximal Cohen–Macaulay module. Although Kodaira vanishing fails in general in characteristic p, certain varieties still satisfy the vanishing conditions required in Proposition 3.1.

Example 3.2. Let k be a perfect field of characteristic $p > 0$. Let Z be either an elliptic curve over k or a $K3$ surface of finite height, and set $X := Z \times \mathbb{P}^1_k$. Let M be an ample line bundle on Z, and set $L := p_1^*M \otimes p_2^*\mathcal{O}_{\mathbb{P}^1_k}(1)$. Then the completion of the section ring $R = R(X, L)$ at its homogeneous maximal ideal admits no module finite Cohen–Macaulay algebra by Bhatt [1, Thm. 1.3, Ex. 3.11]. Moreover, using Kodaira vanishing for K3 surfaces in positive characteristic [8] and Serre duality, together with an argument similar to that of Corollary 3.2, X and L satisfy the hypotheses of Proposition 3.1. Consequently, $R(X, L)_{R_+}$ admits a maximal Cohen–Macaulay module, and hence so does its completion.

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References

[1] B. Bhatt, On the non-existence of small Cohen–Macaulay algebras, *J. Algebra* **411** (2014), 1–11.

[2] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, 2nd ed., Cambridge Stud. Adv. Math., vol. 136, Cambridge Univ. Press, Cambridge, 2013.

[3] A. Grothendieck, Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes, Inst. Hautes Études Sci. Publ. Math. No. 8 (1961), 5–222.

[4] P. Deligne and L. Illusie, Relèvements modulo p^2 et décomposition du complexe de de Rham, *Invent. Math.* **89** (1987), no. 2, 247–270.

[5] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math., vol. 20, Springer, Berlin, 1966.

[6] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math., No. 52, Springer-Verlag, New York–Heidelberg, 1977.

[7] T. Kawasaki, On arithmetic Macaulayfication of Noetherian rings, *Trans. Amer. Math. Soc.* **354** (2002), no. 1, 123–149.

[8] S. Mukai, Counterexamples to Kodaira's vanishing and Yau's inequality in positive characteristics, *Kyoto J. Math.* **53** (2013), no. 2, 515–532.

[9] P. Schenzel, Explicit computations around the Lichtenbaum–Hartshorne vanishing theorem, *Manuscripta Math.* **78** (1993), 57–68.

[10] K. Shimomoto and E. Tavanfar, Remarks on the small Cohen–Macaulay conjecture and new instances of maximal Cohen–Macaulay modules, *J. Algebra* **634** (2023), 667–697.

[11] The Stacks Project Authors, *The Stacks Project*, https://stacks.math.columbia.edu, 2026.

[12] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Stud. Adv. Math., vol. 38, Cambridge Univ. Press, Cambridge, 1994.

[13] L. Xie, On an instance of the small Cohen–Macaulay conjecture, *J. Pure Appl. Algebra* **228** (2024), no. 3, 107491.

[14] L. Xie, Connectedness in Codimension One and the Non-S_2 Locus, arXiv:2605.06617, 2026.

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